Stabilization of the less stable orbit by a tiny near-resonance periodic signal

H. L. Yang and Z. Q. Huang
Institute of Low Energy Nuclear Physics, Beijing Normal University, Beijing 100875, China

E. J. Ding
China Center of Advanced Science and Technology (World Laboratory), P.O. Box 8730, Beijing 100080, China;
Institute of Low Energy Nuclear Physics, Beijing Normal University, Beijing 100875, China;
and Institute of Theoretical Physics, Academia Sinica, Beijing 100080, China

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In this paper, a tiny near-resonance periodic signal is used to stabilize the less stable state under perturbation. Under the action of this external signal, the basin of the period-3 state is extended greatly. Joint use of signal and noise can enhance the stabilization effectiveness of the periodic signal. The sensitive dependence of the effectiveness on signal frequency shows that this stabilization may be due to a certain resonancelike behavior. Numerical calculations also show that the shape of the signal is not an essential factor in our method. This allows a free selection of the applied signal. The method is suitable for driving the system to the less stable state of a pair of coexisting states. [S1063-651X(96)50312-0]

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Multiple stability (i.e., the coexistence of two or more stable states) is a fundamental property of nonlinear systems [1–6]. This means that an ensemble of identical systems starting from different phase points will not be synchronous finally. For a system with a fractal basin boundary, the complex intertwisting of basins makes the final state's dependence on the initial condition very sensitive[3–6]. In this case the prediction of final states is very difficult. For a multiple periodic state, the domains of phase difference make the prediction still more complex. In applications such as laser arrays and Josephson junctions, a collection of nearly identical systems with weak coupling is often desired, however, to evolve in synchronism. Pecora and Carroll first noted this problem [1]. They focused their attention on driving systems starting from points in different domains of a multiple periodic state to be in phase. They added a *pseudoperiodic* signal, which is a chaotic or random signal with some periodic character, instead of the original periodic driving. This method can make almost all orbits starting everywhere in phase. But this method will influence the system parametrically and preparation of the signal is too complex. Recently, Yang and co-workers gave a more flexible approach where random noise was used to drive phase points into synchronism [2]. Despite its simplicity, the method is limited in final state selection since the random noise can only suppress the state less stable under the action of a perturbation. In this paper we show that by adding a tiny near-resonance periodic signal to the system studied, any starting phase point can be driven to a desired attractor, which can be more or less stable under perturbation. Joint use of a periodic signal and a random noise is more effective. Our approach is as simple as Yang's and as effective as Pecora's.

The system we studied is the driven Duffing equation [2]:

$$\frac{d^2x}{dt^2}+0.05\frac{dx}{dt}+x^3=a+b\ \cos t+f_s(t)+f_n(t), \tag{1}$$

where $f_s(t)$ is a tiny near-resonance periodic signal and $f_n(t)$ represents the effect of a random forcing. Throughout this paper, we specifically assume that $f_n(t)$ takes the form $f_n(t)=A_n\eta(t)$, where $\eta(t)$ is a Gaussian white noise with $\langle\eta(t)\rangle=0$ and $\langle\eta(t)\eta(t')\rangle=\delta(t-t')$. Unless pointed out specifically, the signal $f_s(t)$ is chosen to be of the form $f_s(t)=A_s\sin(\omega t+\phi)$. For each initial point, we evolve Eq. (1) for $100T=200\pi$ units of time, $T=2\pi$ being the period of the driving, and then reduce the value of a_n and a_s linearly to zero for another $50T$. All the discussions throughout this paper are based on this point.

For $f_s(t)=f_n(t)=0$, $a=0.15$, and $b=0.21$, Eq. (1) exhibits three coexisting attractors, of period 1, 2, and 3, respectively. Figure 1 shows a part of the intersection of three basins with the phase plane $t\pmod{2\pi}=0$. A uniform grid of 50×50 initial conditions in the rectangle defined by $-0.6<x<0.6$ and $-0.2<dx/dt<0.2$ in the phase plane

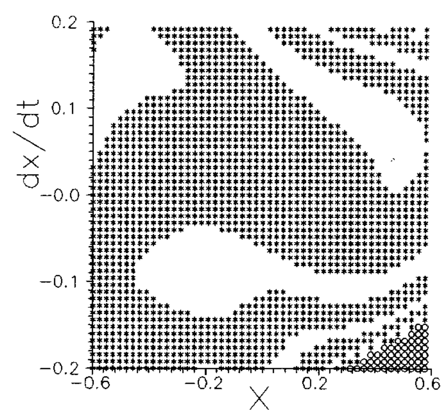

FIG. 1. Part of the basins in the surface of section $t\pmod{2\pi}=0$. It is constructed by starting from 50×50 points uniformly distributed in the rectangle defined by $-0.6<x<0.6, -0.2<dx/dt<0.2$. Parameters are $a=0.15$, $b=0.21$, $A_n=0$, and $A_s=0$. Points evolving to the period 1 and 2 attractor are denoted by ○ and *, respectively. The rest blank region is the basin of the period-3 state.

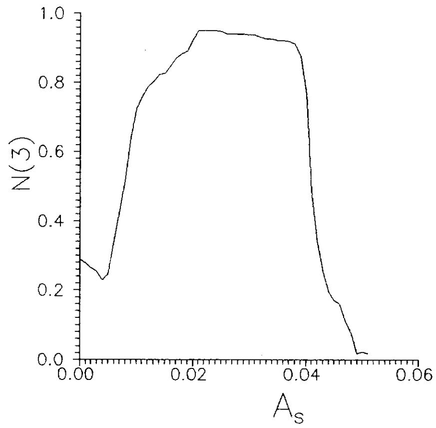

FIG. 2. The fraction of points evolving to the period-3 state vs the increasing of external periodic signal strength. The strength of noise is $A_n=0$. A uniform grid of 30×30 points is used to construct this figure.

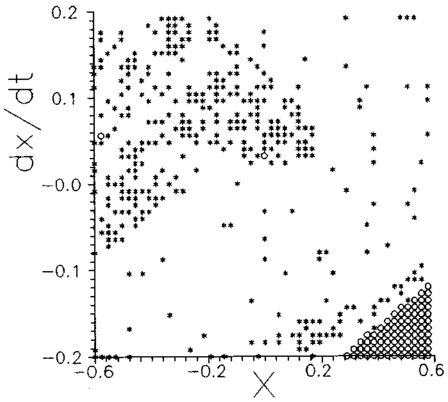

FIG. 3. The basins of each attractor for the system driven by the period-3 signal of strength $A_s=0.01$. The symbols and other parameters are the same as Fig. 1.

$(x,dx/dt,t=0)$ is used to construct this figure. The points evolving to the period 1 and 2 state are denoted by ○ and $*$, respectively. The remaining blank region is the basin of the period-3 state. For the fractal intertwisting of the three attractors' basin, the final state sensitively depends on the phase points where the system starts initially. Furthermore, the using of a random noise can only drive the system to the state more stable under perturbation. The problem of how to drive the system to the period-3 state, which is unstable under perturbation, is still an open question.

Considering that the most stable state of the original system is of identical period to the driving signal, maybe an additional period-3 driving signal can stabilize the period-3 state of the original system. The results of numerical experiment are given below to illustrate this idea.

For $A_n=0.0$, $\omega=0.3333$, and $\phi=0.5$, the effectiveness of the period-3 signal is shown in Fig. 2. It can be seen that with an increase of amplitude A_s of the periodic signal, the fraction of the population of points evolving to the period-3 attractor increases. At about $A_s=0.02$, nearly all of the points have been driven to the desired state. But with further increase of A_s, the number of points to the period-3 attractor decreases. These results confirm that the period-3 signal can really stabilize the period-3 state of the original system. It should be pointed out that the role of the periodic signal here is much different from that in [2]. The signal in that case only destroys the interchange symmetry of the period-m attractor's m domains. But it has no influence on the structure of phase space, in other words, no deformation of the attractor basins or their boundary. It only causes the points originally evolving to different domains of a certain multiple period attractor to be in phase. But the number of the points flowing to different attractors is not changed. In our case, the periodic signal has largely extended the basin of the desired attractor (see Fig. 3) and can drive almost all of the points to the desired state. Also the strength of the applied signal in our case is larger than that in [2].

Considering that use of random noise alone will drive phase points escaping from the period-3 attractor, the result in Fig. 4 is more amazing. Parameters are $w=0.3333$, $\phi=0.5$, and $A_s=0.08$ here. The horizontal axis is the strength of noise while the vertical axis is the fraction of points evolving to the period-3 state. With the increasing of A_n, more and more points flow to the period-3 state. It means that the combined use of a random noise and a periodic signal can enhance the effectiveness of stabilization. And the effectiveness will reach a maximum value at a certain noise strength. This single peak response behavior is just like the stochastic resonance effect.

To test the frequency dependence of our approach, signals of frequencies different from 0.3333 are also applied. Figure 5 shows the numerical results. The vertical axis is the fraction of points evolving to the period-3 attractor, the horizontal axis is the frequency of applied signal. The strengths of the signal and the noise are $A_s=0.01$ and $A_n=0.006$, respectively/ When the external signal is of frequency near 1/3, the output is greatly enhanced. With the deviation from 1/3, the output decreases quickly. This sensitive dependence of the output on signal frequency implies that the stabilization of the period-3 state may be a certain resonance behavior.

To show that this method does not depend on the shape of the signal used, signals with different shapes are applied to our system. For signals of the form

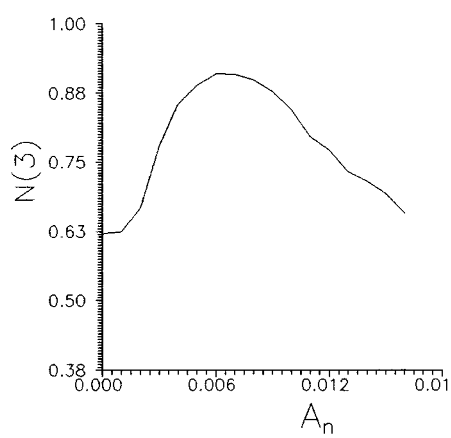

FIG. 4. The fraction of points evolving to the period-3 state vs the increasing of the noise strength. Here the strength of periodic signal $A_s=0.008$. A uniform grid of 30×30 points is used to construct this figure.

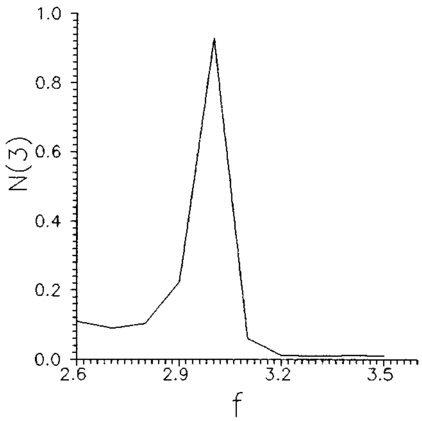

FIG. 5. The fraction of the points flowing to the period-3 state vs the frequency of the applied periodic signal. The strength of noise $A_n=0$. A uniform grid of 30×30 points is used to construct this figure.

$$f_s(t)=\begin{cases} A_s & -\pi\leqslant\omega t+\phi(\mathrm{mod}2\pi)<0 \\ -A_s & 0\leqslant\omega t+\phi(\mathrm{mod}2\pi)<\pi \end{cases} \tag{2}$$

and

$$f_s(t)=\begin{cases} A_s-\dfrac{2A_s}{\pi}(\omega t+\phi) & -\pi\leqslant\omega t+\phi(\mathrm{mod}2\pi)<0 \\ A_s+\dfrac{2A_s}{\pi}(\omega t+\phi) & 0\leqslant\omega t+\phi(\mathrm{mod}2\pi)<\pi, \end{cases} \tag{3}$$

results similar to Fig. 2 are obtained. So we can say that the most essential character of the applied signal is its frequency. Its shape has no essential influence on the results. This nonessential dependence on the signal shape makes its selection more freely. That is important for practical applications.

From above it can be seen that the problem here is somewhat similar to the well known effect ''stochastic resonance'' (ST) [7]. The combined use of noise and periodic signal can enhance the effectiveness of the stabilization. And the effectiveness is dependent on the frequency of the periodic signal. But the two problems have some essential differences. First, the system studied in this paper has its natural frequency (1, 1/2, 1/3 for the three attractors, respectively) without the influence of the external signals (including noise and the periodic signal). But the system in ST has no natural frequency without the external signals. Second, ST is the transformation of energy from noise to signal. It is the behavior of a single system. While the problem here is the transformation of population of phase points from one state to another state, it is the behavior of an ensemble of identical systems. Third, in this paper, the discussions are based on the withdrawal of the external signals after a long enough period of stimulation, while for ST the noise remains for a time as long as the system is operating. For these reasons, we tend to say that the problem here is different from ST.

In Ref. [1], Pecora and Carroll have discussed the problem of driving systems starting from initial points in a different domain of a multiple periodic system in phase. And they have shown that the loss of domains comes about from

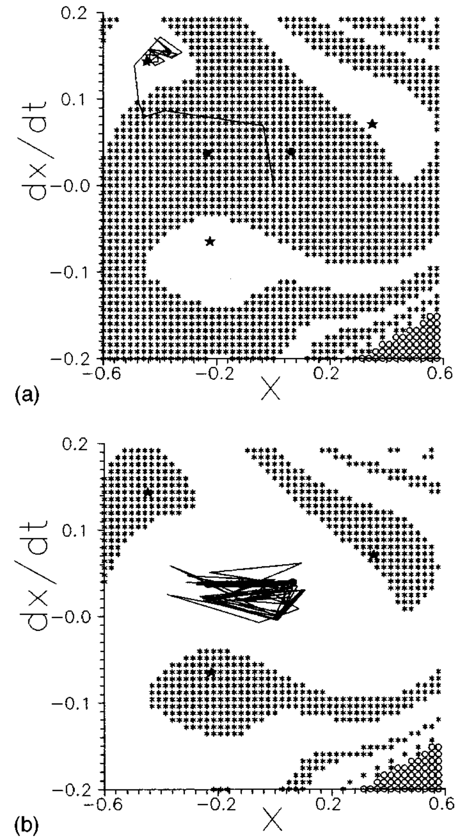

FIG. 6. (a) The typical orbit being attracted to the period-3 state under the external periodic signal which is starting from the basin of the period-2 state. Only one of every three crossing points of the orbit in plane t $(\mathrm{mod}2\pi)=0$ is plotted. The period 2 and 3 states of the original system are denoted by ● and ⋆ respectively. (b) The orbit starting from the basin of the period-2 state is still there under the driving of the external period-3 signal. Others are the same as in (a).

a crisislike behavior. While there is still no explanation about the suppressing of multiple basins, here attempts are made to shed light on this problem for the special case in this paper. (It should be noted that the external signals are withdrawn after a long enough period of time operating.) Figures 6(a) and 6(b) show two orbits starting from different initial points in the surface of section t $(\mathrm{mod}2\pi)=0$. The original system's attractors and their basins are also shown here. The parameters are $A_n=0.0,A_s=0.06$. Orbits shown in Figs. 6(a) and 6(b) are two typical orbits of the system under driving of the external periodic signal. Both of them start from the basin of the original system's period-2 state [8]. In Fig. 6(a), with the action of the external periodic signal, the orbit flows out basin II and evolves to a new appeared attractor. This new attractor is also of period 3, but is slightly different from the original period-3 state. During the linear reduction of external driving, the new attractor loses its stability and the system wanders in basin III. But after the withdrawal of the external signal, it quickly evolves to the period-3 state of the original system. In Fig. 6(b), the orbit under the driving of the external signal is still in basin II, but its behavior has some random character. After the withdrawal of the external signal, it flows to the original period-2 state. Calculations for other initial points show that the new period-3 attractor is stable. From the above numerical results, it can be seen that the mechanism here is not like that for the domain loss in Ref. [1]. There the crisis, or the collision of orbits, that are in different domains of a certain multiple periodic state with an unstable periodic orbit that separates them, causes these orbits to cross the domain boundary and merge into a single one. In our case, the system under driving is still multiple stable, but a new period-3 stable attractor appears. It lies in the original basin of the desired state and has a bigger basin than that of the desired state without external signals. After a long period action of the external signal, a lot of points will be attracted to that new attractor. Since the new attractor lies in the original basin of the desired state, after the withdrawal of the external signal these points attracted to this new attractor will flow to the desired state.

In conclusion, we have shown that with the external driving of a period-3 signal, the perturbation unstable period-3 state can be stabilized. Its basin can be extended greatly. With the joint use of a random noise and a periodic signal, the effectiveness of stabilization can be enhanced. The sensitive dependence on the signal frequency shows that this effect may be due to a certain resonance behavior. The complete understanding of this stabilization might need the systematic investigation of the two-frequency system [9].

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